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LETTER TO THE EDITOR

A conjectured character formula for atypical irreducible modules of the Lie superalgebra sl(m/n)

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Abstract. A formula, equation (17), is proposed for the character of each finite-dimensional irreducible representation of the Lie superalgebra sl(m/n). This new formula appears to cover not only typical but also both singly and multiply atypical representations. A comparison is made with previous conjectures, and the problem of generalising the formula to cover all of the basic classical Lie superalgebras is discussed.

The basic classical Lie superalgebras, G, were classified by Kac (1978) who subdivided them into two types: sl(m/n) and osp($2/2n$) of type I, and osp($2m+1/2n$), osp($2m/2n$) for $m>1$, D($2,1;\alpha$), G(3) and F(4) of type II. Kac also showed that all their finite-dimensional irreducible representations are equivalent to highest weight representations each specified up to isomorphism by means of its highest weight Λ. The corresponding module is denoted by $V(\Lambda)$. Weyl (1926) gave a character formula for all the finite-dimensional irreducible representations of all semisimple Lie algebras. There exists a generalisation of this to the superalgebra case due to Kac (1978) which applies to that class of irreducible representations which are known as typical. Hitherto no generalisation appropriate to all the remaining atypical irreducible representations has been enunciated. However several formulae have been suggested, all with a limited range of validity.

First, Bernstein and Leites (1980) proposed a formula appropriate to sl(m/n) for all m and n. Leites (1980) then proposed that this would also apply to the remaining type-I superalgebras osp($2/2n$) for all n. A more complicated looking formula was then found to apply to singly atypical irreducible representations of osp(m/n) for small values of m and n by Sharp *et al* (1985). More recently, Van der Jeugt (1987) has published a similar formula appropriate to singly atypical irreducible representations of sl(m/n) for all m and n. Furthermore in an unpublished speculation Van der Jeugt (1985) generalised this to cover multiply atypical irreducible representations of sl(m/n), again for all m and n. In seeking to further generalise this result to all the basic classical Lie superalgebras, including those of type II, Cummins (1985) has suggested yet another character formula.

Not all of these formulae are distinct. In particular it can be shown that the most general character formula of Van der Jeugt coincides with that of Bernstein and Leites. However Van der Jeugt takes a much more conservative view of its range of validity. In fact it must be said that checks reveal that none of the character formulae quoted so far cover all the irreducible representations of the superalgebras under consideration.

It is the purpose of this letter to conjecture the validity of a new formula, (17), covering all finite-dimensional irreducible representations of $sl(m/n)$ for all m and n with $m \neq n$. It coincides with that of Kac in the case of all typical irreducible representations and that of Bernstein *et al* in the case of non-low-lying singly and multiply atypical irreducible representations, where low-lying can be given a precise definition in terms of the highest weight vector Λ. The formula is extremely easy to evaluate in terms of characters of the maximal even Lie subalgebra $sl(m)+sl(n)+u(1)$ and our belief in its validity is based on extensive comparison with the explicit results obtained by other methods. In this connection the unpublished tables of Thierry-Mieg (1983) have been of inestimable value, as has the definitive work of Gourdin (1984) applied so extensively to $sl(3/2)$. Comparison has also been made with results based on tensor and Young diagram methods (Dondi and Jarvis 1981, Balantekin and Bars 1982).

After establishing the notation we shall give for comparative purposes the Kac character formula and the Bernstein-Leites-Van der Jeugt character formula, followed by the new conjectured formula, commenting upon the evidence in favour of its validity in the $sl(m/n)$ context and going on to discuss its relevance to other superalgebras.

Let $G = G_{\bar{0}} + G_{\bar{1}}$ be a basic classical Lie superalgebra with even and odd parts $G_{\bar{0}}$ and $G_{\bar{1}}$, respectively. Let Δ_0^+, respectively Δ_1^+, denote the set of positive even, respectively odd, roots of G. Furthermore let

$$\bar{\Delta}_0^+ = \{\alpha \in \Delta_0^+ : \alpha/2 \notin \Delta_1^+\} \tag{1}$$

and

$$\bar{\Delta}_1^+ = \{\beta \in \Delta_1^+ ; 2\beta \notin \Delta_0^+\}. \tag{2}$$

Let the Weyl group of the even subalgebra, $G_{\bar{0}}$, be denoted by W, with elements w of parity $\varepsilon(w)$, and let

$$\rho = \rho_0 - \rho_1 \tag{3}$$

where

$$\rho_0 = \tfrac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha \qquad \text{and} \qquad \rho_1 = \tfrac{1}{2} \sum_{\beta \in \Delta_1^+} \beta. \tag{4}$$

Let Λ be the highest weight of a finite-dimensional irreducible G module $V(\Lambda)$. Both this module and the corresponding representation are said to be *typical* if

$$(\Lambda + \rho, \beta) \neq 0 \qquad \text{for all } \beta \in \bar{\Delta}_1^+ \tag{5}$$

and conversely *atypical* if

$$(\Lambda + \rho, \beta) = 0 \qquad \text{for any } \beta \in \bar{\Delta}_1^+. \tag{6}$$

Refining this terminology in an obvious way, the representation is said to be *singly atypical* if (6) is satisfied by just one $\beta \in \bar{\Delta}_1^+$ and *multiply atypical* if (6) is satisfied by more than one $\beta \in \bar{\Delta}_1^+$. For given highest weight Λ, any odd root $\beta \in \bar{\Delta}_1^+$ satisfying (6) is itself said to be atypical. The module $V(\Lambda)$ is said to be *non-low-lying* if the highest weight Λ is *regular* in the sense that

$$(\Lambda, \alpha) \neq 0 \qquad \text{for all } \alpha \in \bar{\Delta}_0^+ \tag{7}$$

and *low-lying* if

$$(\Lambda, \alpha) = 0 \qquad \text{for any } \alpha \in \bar{\Delta}_0^+. \tag{8}$$

It should be noted that this criterion does not coincide with that applied to sl(m/n) by Van der Jeugt (1987) which involves a considerably more stringent condition than (7).

It is convenient to define

$$L_0 = \prod_{\alpha\in\Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}) \qquad L_1 = \prod_{\beta\in\Delta_1^+} (e^{\beta/2} + e^{-\beta/2}) \tag{9}$$

and

$$L = L_0/L_1. \tag{10}$$

It should be noted that just as $w(L_0) = \varepsilon(w)L_0$ so $w(L) = \varepsilon(w)L$ (Kac 1978) and thus

$$w(L_1) = \prod_{\beta\in\Delta_1^+} (e^{w\beta/2} + e^{-w\beta/2}) = L_1. \tag{11}$$

With this notation, if $V(\Lambda)$ is typical in accordance with the definition (5), then the character of $V(\Lambda)$ has been expressed by Kac (1978) in the form

$$\mathrm{ch}_K V(\Lambda) = L^{-1} \sum_{w\in W} \varepsilon(w)\, e^{w(\Lambda+\rho)}. \tag{12}$$

By virtue of (3), (4), (9), (10) and (11) it follows that for all typical finite-dimensional irreducible representations of all the basic classical Lie superalgebras

$$\mathrm{ch}_K V(\Lambda) = L_0^{-1} \sum_{w\in W} \left(\varepsilon(w)\, e^{w(\Lambda+\rho_0)} \prod_{\beta\in\Delta_1^+} (1 + e^{-w\beta}) \right). \tag{13}$$

The Bernstein–Leites–Van der Jeugt character formula for both typical and atypical finite-dimensional irreducible representations of all the type-I basic classical Lie superalgebras takes the very similar form

$$\mathrm{ch}_J V(\Lambda) = L_0^{-1} \sum_{w\in W} \left(\varepsilon(w)\, e^{w(\Lambda+\rho_0)} \prod_{\beta\in\Delta_r^+} (1 + e^{-w\beta}) \right) \tag{14}$$

where

$$\Delta_r^+ = \Delta_1^+ \backslash \bar{\Delta}_a^+ \tag{15}$$

with

$$\bar{\Delta}_a^+ = \{\beta \in \bar{\Delta}_1^+ : (\Lambda + \rho, \beta) = 0\}. \tag{16}$$

Whilst this formula is correct in the case of all typical representations, for which it is identical to the Kac formula (13), and appears to be correct in the case of all singly atypical representations of type-I superalgebras, it is certainly not true for all of the multiply atypical representations which arise in the case of sl(m/n) for m and n sufficiently large. In particular, comparison with the tables of Thierry-Mieg (1983) indicates seven cases of disagreement for sl(2/3) and eleven cases for sl(2/4). This discrepancy is confirmed by the work of Gourdin (1984) which spells out in a series of tables the characters of all the finite-dimensional irreducible representations of sl(3/2); it is found that (14) breaks down for $\{0,0|0|1\}$ and for both $\{a,0|0|0\}$ and $\{0,a|-1-a|0\}$ with $a = 0, 1, 2, \ldots$. There can be no doubt about this limitation on the range of validity of (14) since the above list includes the identity representation, whilst all the others correspond to purely covariant or purely contravariant tensor representations for which the characters are well known (Dondi and Jarvis 1981, Balantekin and Bars 1982).

In an effort to find some modification of (14) which covers correctly all the irreducible representations of sl(3/2), a REDUCE program was used to explore a variety of possibilities. This exploration, later extended to $\mathrm{sl}(m/n)$ with larger values of m and n, led to the following.

Conjecture. If $V(\Lambda)$ is a finite-dimensional irreducible module of $\mathrm{sl}(m/n)$ with $m \neq n$ then the character of $V(\Lambda)$ is given by

$$\mathrm{ch}_H V(\Lambda) = L_0^{-1} \sum_{w \in W} \left(\varepsilon(w)\, \mathrm{e}^{w(\Lambda+\rho_0)} \prod_{\beta \in \Delta_s^+} (1 + \mathrm{e}^{-w\beta}) \right) \tag{17}$$

where the final product is taken over the set of positive odd roots

$$\Delta_s^+ = \Delta_1^+ \backslash (\bar{\Delta}_a^+ \cup \bar{\Delta}_b^+) \tag{18}$$

with $\bar{\Delta}_a^+$ defined to be the set of atypical positive odd roots specified by (16) and with $\bar{\Delta}_b^+$ defined to be an additional set of positive odd roots specified by

$$\bar{\Delta}_b^+ = \{\beta \in \bar{\Delta}_1^+ \backslash \bar{\Delta}_a^+ : \beta = \beta' + \alpha' \text{ with } \beta' \in \bar{\Delta}_a^+,\ \alpha' \in \bar{\Delta}_0^+ \text{ and } (\Lambda, \alpha') = 0\}. \tag{19}$$

This conjectured character formula (17) reduces to that of Kac, (13), in the case of all the typical irreducible representations of $\mathrm{sl}(m/n)$, and to that of Bernstein–Leites–Van der Jeugt, (14), in the case of all non-low-lying singly and multiply atypical irreducible representations of $\mathrm{sl}(m/n)$. Moreover for the identity module $V(0)$, the set Δ_s^+ is empty and (17) reduces to the Weyl character formula for the identity module of the Lie algebra $G_{\bar{0}}$.

An exhaustive comparison with all the sl(3/2) irreducible representations studied by Gourdin (1984), whether typical, singly atypical, multiply atypical, non-low-lying or low-lying yields no discrepancies save for one or two arising from what are self-evidently misprints in the tables of Gourdin. In addition (17) has been well tested against the tabulation of Thierry-Mieg (1983) for all the superalgebras $\mathrm{sl}(m/n)$ with $m+n \leqslant 6$. Finally a large number of checks have been made by comparing (17) with results obtained by applying tensor and Young diagram methods to $\mathrm{sl}(m/n)$ for various m and n such as $m=6$ and $n=4$ in the case of purely covariant or purely contravariant tensor irreducible representations. In such cases the character formula is known (Dondi and Jarvis 1981, Balantekin and Bars 1982, King 1983, Berele and Regev 1987) and the software package SCHUR was used to make the checks. Perfect agreement was obtained.

It might be thought that the conjecture (17) would also apply to the remaining type-I superalgebras $\mathrm{osp}(2/2n)$. This is not the case. It is found that discrepancies with the results of Thierry-Mieg (1983) occur in the case of the irreducible representations (1; 0, 0) of osp(2/4) and (1; 0, 0, 0), (2; 0, 0, 0) and (2; 1, 0, 0) of osp(2/6). Remarkably, agreement is obtained in all the other forty or more atypical cases that have been tested. More important, however, is the fact that this is a failure at the singly atypical level. At this level the Bernstein–Leites–Van der Jeugt formula (15) produces no discrepancies with known results for any of the type-I superalgebras $\mathrm{sl}(m/n)$ or $\mathrm{osp}(2/2n)$.

Since $\mathrm{osp}(2/2n)$ possesses no multiply atypical irreducible representations, the situation for type-I superalgebras appears to be such that all those irreducible representations which are typical are covered by (13), all which are singly atypical are covered by (14) and all which are multiply atypical are covered by (17). An alternative summary of the situation is that all the irreducible representations of sl(m/n) with $m \neq n$ are covered by (17) and all those of osp$(2/2n)$ by (14).

Turning to type-II superalgebras, a number of possibilities present themselves. In attempting to extend the Bernstein–Leites–Van der Jeugt result (14) to cover the case of all typical and atypical irreducible representations of all the basic classical Lie superalgebras, both type I and type II, Cummins (1985) suggested the proposition that

$$\mathrm{ch}_C V(\Lambda) = L_0^{-1} \sum_{w \in W} \left(\varepsilon(w)\, \mathrm{e}^{w(\Lambda+\rho_0)} \prod_{\beta \in \Delta_t^+} (1 + \mathrm{e}^{-w\beta}) \right) \tag{20}$$

where

$$\Delta_t^+ = \Delta_1^+ \backslash \Delta_c^+ \tag{21}$$

with

$$\Delta_c^+ = \{\beta \in \Delta_1^+ : (\Lambda + \rho, \beta) = (\beta, \beta)/2\}. \tag{22}$$

The most promising generalisation of the conjecture (17) is provided by

$$\mathrm{ch}_G V(\Lambda) = L_0^{-1} \sum_{w \in W} \left(\varepsilon(w)\, \mathrm{e}^{w(\Lambda+\rho_0)} \prod_{\beta \in \Delta_g^+} (1 + \mathrm{e}^{-w\beta}) \right) \tag{23}$$

where now

$$\Delta_g^+ = \Delta_1^+ \backslash (\Delta_c^+ \cup \Delta_d^+) \tag{24}$$

with Δ_c^+ defined by (22) and Δ_d^+ defined by

$$\Delta_d^+ = \{\beta \in \Delta_1^+ \backslash \Delta_c^+ : \beta = \beta' + \alpha' \text{ with } \beta' \in \Delta_c^+, \alpha' \in Q_+, \alpha' \neq 0 \text{ and } (\Lambda, \alpha') = 0\} \tag{25}$$

where

$$Q_+ = \sum_{\alpha \in \Delta_0^+} \mathbb{Z}_+ \alpha \qquad \text{with} \qquad \mathbb{Z}_+ = \{0, 1, 2, \ldots\}. \tag{26}$$

These possibilities can be tested by consideration of the type-II superalgebra G(3) for which $\bar{\Delta}_1^+ \neq \Delta_1^+$ and which possesses singly atypical irreducible representations. These tests reveal that (14) disagrees with the results of Thierry-Mieg (1983) in thirteen cases, whilst (20) disagrees in only eight cases. Moreover in seven of these the character given by (20) is exactly twice the correct result. Finally (23) disagrees in only two cases, namely (3; 0, 0) and (5; 0, 0). Whilst this is not entirely satisfactory it should be pointed out that although (23) and (20) do not coincide with Kac's formula (13) they both seem to yield the correct characters of all typical representations of G(3).

Multiply atypical representations of type-II superalgebras occur in the case of $F(4)$. Here because $\bar{\Delta}_1^+ = \Delta_1^+$, (14) and (20) now coincide. It is found that as in the case of G(3), (14) in some cases yields twice the correct result and in others yields a character which is the sum of two inequivalent irreducible characters. The conjecture (23) is somewhat better with a discrepancy occurring in only 4 cases out of the 25 tabulated by Thierry-Mieg (1983).

Finally application was made to the type-II superalgebra D(2, 1; 1) = osp(4/2). Here comparison with the tabulation of Thierry-Mieg (1983) and the more complete results of Farmer and Jarvis (1984) indicates that (14) fails only for the identity representation (0; 0, 0) and the adjoint representation (2; 0, 0), whilst (23) fails only for the adjoint representation. Unfortunately it has to be admitted that a computer search has revealed that no formula of the type (23) with any choice of Δ_g^+ as a subset of Δ_1^+ fits the known character of the adjoint representation of osp(4/2).

To date we have not attempted to prove the validity of the conjecture (17) for all irreducible representations of sl(m/n) nor have we established the range of validity of the more general formula (23). Even though the latter is not true in all cases it certainly covers an extraordinarily wide range of cases. As such it appears to be worthy of further study. This is particularly the case because it is so easy to use in the explicit calculation of superalgebra characters. In practice a number of hand calculations augmented by computer calculations using PASCAL have all been based on the exploitation of (14), (17), (20) and (23) in the form suggested by Cummins (1985):

$$\mathrm{ch}\ V(\Lambda)=\sum_{\boldsymbol{m}}\chi^{\Lambda-\boldsymbol{m}\cdot\boldsymbol{\beta}} \tag{27}$$

where $\boldsymbol{\beta}=(\beta_1,\beta_2,\ldots,\beta_d)$ and $\boldsymbol{m}=(m_1,m_2,\ldots,m_d)$ with $\beta_i\in\Delta_g^+$ for $i=1,2,\ldots,d$ and $d=|\Delta_g^+|$. The summation is carried out over all those vectors $\boldsymbol{m}$ whose components m_i for $i=1,2,\ldots,d$ take on the values 0 or 1. The notation is such that

$$\chi^{\Lambda-\boldsymbol{m}\cdot\boldsymbol{\beta}}=\begin{cases}\varepsilon(w)\chi^{\lambda} & \text{if } w(\Lambda-\boldsymbol{m}\cdot\boldsymbol{\beta}+\rho_0)-\rho_0=\lambda \quad \text{for some } w\in W\\ 0 & \text{if } w(\Lambda-\boldsymbol{m}\cdot\boldsymbol{\beta}+\rho_0)-\rho_0=-(\Lambda-\boldsymbol{m}\cdot\boldsymbol{\beta}) \quad \text{for any } w\in W\end{cases} \tag{28}$$

where λ is G-dominant in the sense that $w(\lambda+\rho_0)-\rho_0\leq\lambda$ for all $w\in W$, and χ^λ is the character of the irreducible representation of the even subalgebra $G_{\bar{0}}$ with highest weight given by the Weyl character formula (Weyl 1926)

$$\chi^{\lambda}=L_0^{-1}\sum_{w\in W}\mathrm{e}^{w(\lambda+\rho_0)}. \tag{29}$$

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References

Balantekin A B and Bars I 1982 *J. Math. Phys.* **23** 1239-47
Berele A and Regev A 1987 *Adv. Math.* **64** 118-75
Bernstein I N and Leites D A 1980 *C.R. Acad. Bulg. Sci.* **33** 1049-51
Cummins C J 1985 *A character formula for atypical representations of basic classical Lie superalgebras* University of Southampton (unpublished)
Dondi P H and Jarvis P D 1981 *J. Phys. A: Math. Gen.* **14** 547-63
Farmer R J and Jarvis P D 1984 *J. Phys. A: Math. Gen.* **17** 2365-87
Gourdin M 1984 *Naturally atypical supertableaux of the superunitary group SU(3/2)* Université Pierre et Marie Curie, Paris PAR LPTHE 84/26
Kac V 1978 *Lecture Notes in Mathematics* vol 676 (Berlin: Springer) pp 597-626
King R C 1983 *Ars. Comb.* **16A** 269-87
Leites D A 1980 *C.R. Acad. Bulg. Sci.* **33** 1053-5
Sharp R T, Van der Jeugt J and Hughes J W B 1986 *J. Math. Phys.* **26** 901-12
Thierry-Mieg J 1983 *Table des representations irreducibles des superalgebres de Lie SU(m/n), SU(n/n)/U(1), $Os_p(m/n)$, D(2/1; α), G(3), F(4)* (unpublished)
Van der Jeugt J 1985 *Structure of atypical representations of the Lie superalgebras spl(m, n)* Rijksuniversiteit Gent (unpublished)
—— 1987 *J. Phys. A: Math. Gen.* **20** 809-24
Weyl H 1926 *Math. Z.* **24** 377-95